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Approximate solutions of the Gołąb–Schinzel equation

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Abstract

We determine all unbounded approximate solutions of the Gołąb–Schinzel functional equation in the class of functions continuous at 0.

1. Introduction

The Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the unknown function, is one of the most important composite type functional equations. Some information concerning (1), recent results, applications and numerous references one can find in [1–5,8]. In [6] the problem of the Hyers–Ulam stability of (1) has been considered. It has been proved there that in the class of continuous functions Eq. (1) is superstable, i.e. every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$|f(x + f(x)y) - f(x)f(y)| \leqslant \varepsilon \quad \text{for } x, y \in \mathbb{R}, \tag{2}$$

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where ε is a fixed positive real number, is either bounded or is a solution of (1). For more information concerning superstability we refer to [9] (Chapter 5).

It is known (cf. [7]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of Eq. (1) we have the product, but in (2) we measure the distance between the two sides of (1) using the difference. Therefore, it is more natural to measure the difference between 1 and the quotients of the sides of Eq. (1). In [7] it has been proved that for the exponential equation this approach leads to the traditional stability.

In the present paper we show that in the case of the Gołąb–Schinzel equation, the situation is different.

2. Results

Theorem 1. *Let* $\varepsilon \in (0, 1)$. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous at* 0 *solution of the system of conditional functional inequalities*:

$$\left| \frac{f(x + f(x)y)}{f(x)f(y)} - 1 \right| \leqslant \varepsilon \tag{3}$$

for $x, y \in \mathbb{R}$ *such that* $f(x)f(y) \neq 0$; *and*

$$\left| \frac{f(x)f(y)}{f(x + f(x)y)} - 1 \right| \leqslant \varepsilon \tag{4}$$

for $x, y \in \mathbb{R}$ *such that* $f(x + f(x)y) \neq 0$. *Then either*

$$f(x) \in \left[\frac{1}{1+\varepsilon}, 1 + \varepsilon \right] \textit{ for } x \in \mathbb{R}; \tag{5}$$

or f has one of the forms

$$f(x) = 0 \textit{ for } x \in \mathbb{R}, \tag{6}$$

$$f(x) = 1 + cx \textit{ for } x \in \mathbb{R}, \tag{7}$$

$$f(x) = \max\{1 + cx, 0\} \textit{ for } x \in \mathbb{R}, \tag{8}$$

where c is a non-zero real constant.

Proof. Let

$$F_f := \{x \in \mathbb{R} : f(x) = 0\}.$$

Furthermore, for a fixed $x \in \mathbb{R}$, let $g_x : \mathbb{R} \to \mathbb{R}$ be given by

$$g_x(y) = x + f(x)y \quad \text{for } y \in \mathbb{R}. \tag{9}$$

Since $\varepsilon \in (0, 1)$, from (3) and (4) it results that $g_x(y) \in F_f$ if and only if $x \in F_f$ or $y \in F_f$. Therefore, for every $x \in \mathbb{R} \setminus F_f$, the set F_f is strongly invariant under g_x, i.e.

$$g_x(y) \in F_f \text{ if and only if } y \in F_f. \tag{10}$$

If $F_f = \emptyset$, then (3) and (4) occur for all $x, y \in \mathbb{R}$. Hence

$$\left|\frac{1}{f(x)} - 1\right| = \left|\frac{f(x + f(x)\frac{x}{1-f(x)})}{f(x)f(\frac{x}{1-f(x)})} - 1\right| \leqslant \varepsilon$$

and

$$|f(x) - 1| = \left|\frac{f(x)f(\frac{x}{1-f(x)})}{f(x + f(x)\frac{x}{1-f(x)})} - 1\right| \leqslant \varepsilon$$

for all $x \in \mathbb{R}$ with $f(x) \neq 1$. This implies (5).

It is easily seen that $F_f = \mathbb{R}$ implies (6). So it remains to consider the situation where F_f is a non-empty proper subset of $\mathbb{R}$. In this case, taking in (4) an $x \in \mathbb{R} \setminus F_f$ and $y = 0$, we obtain that $|f(0) - 1| \leqslant \varepsilon$, which means that $f(0) \geqslant 1 - \varepsilon > 0$. Hence, as f is continuous at 0, there exists an open neighbourhood U of 0 such that $U \subset \mathbb{R} \setminus F_f$. Fix an $x \in \mathbb{R} \setminus F_f$. Then g_x is a homeomorphism and $g_x(0) = x$. Thus $g_x(U)$ is an open neighbourhood of x. Moreover, in view of (10), $g_x(U) \subset \mathbb{R} \setminus F_f$. Consequently $\mathbb{R} \setminus F_f$ is open, so F_f is a closed set. Let $F_f^- := (-\infty, 0] \cap F_f$ and $F_f^+ := [0, \infty) \cap F_f$. Obviously at least one of the sets F_f^- and F_f^+ is non-empty. Suppose that $F_f^- \neq \emptyset$ and $F_f^+ \neq \emptyset$. Let $z_1 := \max F_f^-$ and $z_2 := \min F_f^+$. Then $z_1 < 0 < z_2$ and

$$(z_1, z_2) \cap F_f = \emptyset. \tag{11}$$

Since f is continuous at 0 and $f(0) > 0$, there is an $x \in (0, z_2)$ with $f(x) > 0$. If $f(x) \in (0, 1)$, we have $z_1 < x + z_1 < x + z_1 f(x) < x$. Hence $g_x(z_1) \in (z_1, x) \subset (z_1, z_2)$. On the other hand, by (10) $g_x(z_1) \in F_f$, which contradicts (11).

If $f(x) \geqslant 1$, then using (10), we obtain $F_f \ni {g_x}^{-1}(z_2) = \frac{z_2 - x}{f(x)} \in (0, z_2) \subset (z_1, z_2)$. This again yields a contradiction with (11). Therefore we have proved that exactly one of the sets F_f^- and F_f^+ is non-empty. Since the proof in both cases is analogous, assume that $F_f^- \neq \emptyset$ and $F_f^+ = \emptyset$. Let

$$z := \max F_f < 0. \tag{12}$$

Fix an $x \in (z, \infty)$. By (12), $f(x) \neq 0$. If $f(x) < 0$, then using (10) and (12), we obtain $z < x < x + f(x)z = g_x(z) \leqslant z$, which brings a contradiction. Therefore $f(x) > 0$, so g_x is strictly increasing and, in view of (10) and (12), we get that $g_x(z) \leqslant z$ and ${g_x}^{-1}(z) \leqslant z$. Hence $g_x(z) = z$, which implies that $f(x) = 1 - \frac{x}{z}$. In this way we have proved that

$$f(x) = 1 - \frac{x}{z} \quad \text{for } x \in [z, \infty). \tag{13}$$

Now, suppose that there is an $x < z$ with $f(x) \neq 0$. If $f(x) > 0$, then according to (10), $F_f \ni {g_x}^{-1}(z) = \frac{z - x}{f(x)} > 0$, which contradicts (12). Consequently $f(x) < 0$, g_x is strictly decreasing and, in virtue of (10) and (12), we obtain

$$(-\infty, g_x(z)) \cap F_f = g_x((z, \infty)) \cap F_f = \emptyset.$$

Thus $g_x(z) \leqslant z$ and $F_f \subset [g_x(z), z]$. Since F_f is closed, this means that there exists

$$z_0 := \min F_f. \tag{14}$$

As $-z \in (z, \infty) \subset \mathbb{R} \setminus F_f$, using (10) and (13), we obtain

$$F_f \ni g_{(-z)}(z_0) = -z + f(-z)z_0 = -z + \left(1 - \frac{-z}{z}\right) z_0 = -z + 2z_0.$$

Hence, by (14) $z_0 \leqslant -z + 2z_0$, so $z \leqslant z_0$. Thus $z_0 = z$ and $F_f = \{z\}$. Therefore, we have proved that either $F_f = (-\infty, z]$ or $F_f = \{z\}$.

If the first possibility occurs, then using (13), we obtain that *f* has the form (8) with $c := -\frac{1}{z} \neq 0$. If the second one holds, then according to (10), we get that

$$g_x(z) = z \quad \text{for } x \in \mathbb{R} \setminus F_f = \mathbb{R} \setminus \{z\}.$$

Hence

$$f(x) = 1 - \frac{x}{z} \quad \text{for } x \in \mathbb{R} \setminus \{z\}.$$

Since $f(z) = 0$, this implies that *f* has the form (7) with $c := -\frac{1}{z} \neq 0$, which completes the proof. □

The following example shows that the continuity of *f* at 0 is an essential assumption in Theorem 1.

Example 1. Let $p \in \mathbb{R} \setminus \{0\}$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 - \frac{x}{p} & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then *f* is continuous at *p* and satisfies the system (3)–(4) (in fact *f* is a solution of (1)). However, neither (5) holds nor *f* has one of the forms (6)–(8).

Remark 1. Note that every function of the form (6)–(8) satisfies (1). Then from Theorem 1 it follows that in the class of functions continuous at 0, Eq. (1) is superstable in the sense that every function satisfying (3) and (4) either is “close” to 1 or is a solution of (1).

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